Random Variables

Charlie Gibbons
University of California, Berkeley

ARE 210

Fall 2015

Notes

Outline

Notes

Random variables

A *random variable* is a mapping from the sample space to the real line.

Example: Flipping a coin 10 times has 2^{10} elements in the sample space. We can simplify this by letting $X =$ number of heads.

Notes

Cumulative distribution function

A random variable is associated with a *cumulative distribution function* $\Pr(X \leq x) \equiv F(x)$ for all $-\infty < x < \infty$.

Properties of the CDF:

- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to \infty} F(x) = 1$
- $F(x)$ is non-decreasing; $F'(x) \geq 0$

Notes

Example

Suppose that we toss a coin 4 times. Find the CDFs of

- $X =$ the number of heads
- $Y =$ the number of tails

Notes

Continuous versus discrete

A random variable is *continuous* if $F(x)$ is a continuous function of x.

A random variable is *discrete* if $F'(x) = 0$ almost everywhere; the CDF follows a stairstep pattern.

A random variable can be a mixture of these two.

Notes

Probability functions

Discrete random variables have *probability mass functions* (PMFs): $f(x) = \Pr(X = x)$.

Continuous random variables have *probability density functions* (PDFs): $f(x) = F'(x)$. Note that, for these random variables, $\Pr(X = x) = 0$ for all x.

Consider: For a given random variable, is the PMF unique? The PDF?

Notes

CDF for discrete and continuous cases

The CDF is

$$F(x) = \begin{cases} \sum_{-\infty}^{x} \Pr(X = x) \text{ for discrete random variables} \\ \int_{-\infty}^{x} f(x)\, dx \text{ for continous random variables} \end{cases}$$

Notes

Joint PMF example

The *joint probability mass function (joint PMF)*, $f_{X,Y}$ is

$$f_{X,Y}(x, y) = \Pr(X = x \text{ and } Y = y).$$

What is $f_{X,Y}(6, 5)$?

x, y	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Notes

Joint CDF definition

The *joint cumulative distribution function* (joint CDF), $F_{X,Y}(x, y)$, of the random variables X and Y is defined by

$$F_{X,Y}(x,y) = \Pr(X \leq x \text{ and } Y \leq y)$$
$$= \begin{cases} \sum_{s=-\infty}^{x} \sum_{t=-\infty}^{y} f_{X,Y}(s,t) \text{ for the discrete case or} \\ \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(s,t)\, ds\, dt \text{ for the continuous case.} \end{cases}$$

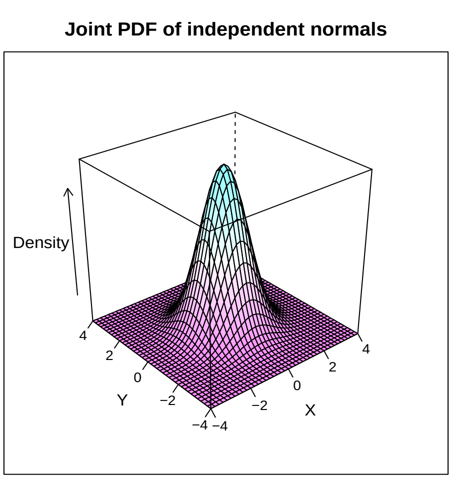
Notes

Joint CDF example

What is $F_{X,Y}(2,3)$?

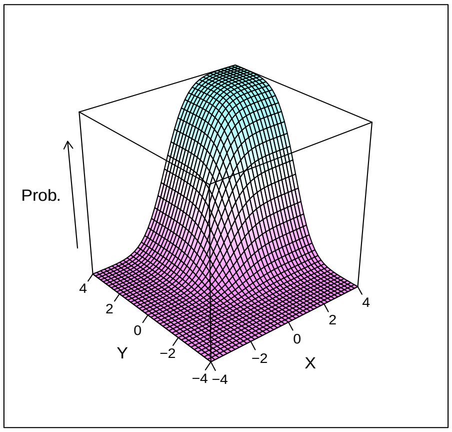
x, y	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Notes



Notes

Joint CDF of independent normals



Notes

Marginal distributions

The joint distribution gives the probability/density of a pair of (x, y). Often, we may be interested in just the probability of x for any value of y; this is the *marginal distribution* of x.

For discrete X and Y, the marginal distribution of X is

$$\Pr(x) = \sum_{y=-\infty}^{\infty} f_{X,Y}(x, y).$$

For continuous X and Y, the marginal distribution of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy.$$

Notes

Uniqueness

Given a joint distribution between X and Y, the marginal distributions F_X and F_Y are uniquely determined.

Given the marginal distributions of X and Y, the joint distribution is *not* uniquely determined.

Notes

Independence

Two random variables are independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

The set of random variables $X_1, \dots, X_N$ are *independent and identically distributed (iid)* if

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1)\cdots f_{X_N}(x_N) = f_X(x_1)\cdots f_X(x_N)$$

Notes

$Y = F(X)$

Let X be a continuous random variable with CDF $F(x)$. Let $Y = F(X)$.

Is Y a random variable?

What is the distribution of Y?

We have

$$\begin{aligned}\Pr(Y \leq y) &= \Pr(F(X) \leq y) \\ &= \Pr(X \leq F^{-1}(y)).\end{aligned}$$

Notes

Quantile function

What is $F^{-1}(y)$?
It is the *quantile function*; it answers, what is the value of x such that $F(x) = y$—that y% of the time, X falls below x. Let $F(x^*) = y$.

This gives

$$\begin{aligned}\Pr(X \leq F^{-1}(y)) &= \Pr\left(X \leq F^{-1}\left(F(x^*)\right)\right) \\ &= \Pr(X \leq x^*) \\ &= y.\end{aligned}$$

Notes

Distribution of the CDF

In summary, for $y \in [0,1]$, $\Pr(Y \leq y) = y$.

The CDF of *any* continuous random variable X is itself a *uniform random variable* on the range $[0,1]$.

Notes

Going the other direction

Let $U \sim \text{Uniform}[0,1]$. Let X be a continuous random variable such that $X \sim F$. Then $F^{-1}(U) \sim F(X)$.

Why do we care?
If we can generate a uniform random variable and we can find the inverse of the CDF of X, *we can generate the random variable* X.

Notes

Generalization

Let X be a continuous random variable and let $r(x)$ be a strictly monotonic function on the support of X.

$y = r(x)$ has an inverse, $x = s(y)$.

We have

$$f_Y(y) = f_X(s(y)) \left| \frac{ds(y)}{dy} \right| .$$

Notes

Application to previous example

We can apply this to our previous example, giving

$$f_Y(y) = f_X\left(F^{-1}(y)\right)\left|\frac{dF^{-1}(y)}{dy}\right|.$$

Recall that

$$\left[g^{-1}\right]'(a) = \frac{1}{g'\left(g^{-1}(a)\right)}.$$

In our case, $g(\cdot)$ is $F(\cdot)$ and $g'(\cdot) = f_X(\cdot)$.

Notes

Uniform PDF

This gives:

$$\begin{aligned} f_Y(y) &= f_X\left(F^{-1}(y)\right)\left|\frac{1}{f_X\left(F^{-1}(y)\right)}\right| \\ &= 1, \end{aligned}$$

which is the PDF for a uniform random variable on the interval $[0, 1]$.

Notes

Sums of random variables

Consider the random variable $Z = X + Y$. We can find the distribution of Z according to

$$\Pr(Z = z) = \sum_{s=-\infty}^{\infty} \Pr(X = s, Y = z - s)$$

for a discrete random variable and

$$f(Z = z) = \int_{-\infty}^{\infty} f(X = s, Y = z - s)\, ds$$

for a continuous random variable.

Notes

Distribution of the maximum

Let $X_1, \dots, X_N \sim F$. Let $Z = \max\{X_1, \dots, X_N\}$. Find the distribution of Z.

Notes

CDF

To find a distribution, it is often easiest to start by finding the CDF:

$$\begin{aligned} F_Z(z) &= \Pr(Z \leq z) \\ &= \Pr(\max\{X_1, \dots, X_N\} \leq z) \\ &= \Pr(X_1 \leq z, \dots X_N \leq z) \\ &= \Pr(X_1 \leq z) \cdots \Pr(X_N \leq z) \\ &= F(z)^N. \end{aligned}$$

Notes

PDF

To find the PDF, we take the derivative with respect to z:

$$\begin{aligned} f_Z(z) &= \frac{dF_Z(z)}{dz} \\ &= \frac{dF_X(z)^N}{dz} \\ &= NF_X(z)^{N-1} f_X(z). \end{aligned}$$

Notes