Autoregressive Models

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Notes

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Notes

Time series

A *time series* is a set of observations from a single unit measured at equidistant periods in time.

Often, the past influences the future. This dependence requires a model.

The goal of time series analysis is typically predicting the future, rather than explaining the past.

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Stationarity

Before working with a time series, we need to assume that our *stochastic process* is *(covariance) stationary*, with the properties:

$$\mathbb{E}(y_t) = \mu \quad \text{(time-invariant mean)}$$
$$\text{Var}(y_t) = \sigma^2 \quad \text{(time-invariant variance)}$$
$$\text{Cor}(y_t, y_{t-s}) = \text{Cor}(y_t, y_{t+s}) = \gamma(s)$$

$\gamma(s)$ is called the *autocorrelation function*.

Stationarity essentially means that probability distributions are stable over time; y_t are identically distributed, though not independent.

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Why stationarity?

Why is stationarity important?

If we say that y_t can change in an arbitrary way (*i.e.*, one where the probability distribution is not stable over time), then how can we learn about patterns?

Put another way, econometrics is about using data to learn about means. If each point in time has a different mean, we can never have multiple data points to average to estimate that particular mean. We need to assume a constant mean so that we can take averages and get meaningful results.

Additionally, regressions involving non-stationarity variables often lead to incorrect *spurious* results.

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Weak dependence

To the assumption of stationarity, we must add the assumption of *weak dependence* or *asymptotic uncorrelation*: as s increases, $\gamma(s)$ goes to 0.

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Example

Suppose that y_0 could be -1 or 1 with equal probability and that $y_s = (-1)^s y_0$. Then we have:

$$\mathbb{E}[y_0] = \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$$
$$\mathbb{E}[y_t] = \mathbb{E}[\mathbb{E}[y_t \mid y_0]] = \mathbb{E}[(-1)^t y_0] = (-1)^t \mathbb{E}[y_0] = 0$$
$$\text{Var}(y_t) = \frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2 = 1$$
$$\text{Cor}(y_t, y_{t+s}) = \text{Cor}(y_t, y_{t-s}) = \begin{cases} 1 & \text{if } s \text{ is even} \\ -1 & \text{if } s \text{ is odd} \end{cases}$$

This process is stationary, but not weakly dependent.

Notes

Autoregressive model

Let

$$y_t = \phi y_{t-1} + \nu_t;$$

this is an *autoregressive model of order 1*—AR(1).

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White noise

ν_t is called *white noise* with the properties:

- $\text{Cov}(\nu_t, \nu_{t-s}) = 0 \, \forall s \neq 0$ (these errors are uncorrelated)
- $\mathbb{E}(\nu) = 0$ (have mean 0)
- $\text{Var}(\nu_t) = \sigma_\nu^2$ (time-invariant variance)

This term is often called an "innovation," "shock," or "news" since it is not known in the previous periods and is not correlated with future shocks. Put another way, the white noise error must be uncorrelated with all past and future values of ν.

Notes

AR(1) as an infinite sum

Let's rewrite the AR(1) model as:

$$\begin{aligned} y_t &= \phi(\phi y_{t-2} + \nu_{t-1}) + \nu_t \\ &= \phi(\phi^2 y_{t-3} + \phi\nu_{t-2} + \nu_{t-1}) + \nu_t \\ &= \ldots \\ &= \sum_{s=0}^{\infty} \phi^s \nu_{t-s} \end{aligned}$$

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Expected value

Now we can check the requirements of stationarity.

$$\begin{aligned} \mathbb{E}[y_t] &= \mathbb{E}\left[\sum_{s=0}^{\infty} \phi^s \nu_{t-s}\right] \\ &= \sum_{s=0}^{\infty} \phi^s \mathbb{E}[\nu_{t-s}] \\ &= 0. \end{aligned}$$

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Variance

And the variance is:

$$\begin{aligned} \mathrm{Var}(y_t) &= \mathrm{Var}\left(\sum_{s=0}^{\infty} \phi^s \nu_{t-s}\right) \\ &= \sum_{s=0}^{\infty} \phi^{2s} \mathrm{Var}(\nu_{t-s}) \\ &= \sigma_\nu^2 \sum_{s=0}^{\infty} \left(\phi^2\right)^s \\ &= \frac{\sigma_\nu^2}{1-\phi^2}. \end{aligned}$$

assuming that $|\phi| < 1$.

Notes

Covariance

The covariance between two periods is:

$$\begin{aligned}\text{Cov}(y_t, y_{t-s}) &= \text{Cov}\left(\phi^s y_{t-s} + \sum_{r=0}^{s-1} \phi^r \nu_{t-r}, y_{t-s}\right) \\ &= \phi^s \text{Var}(y_{t-s})\end{aligned}$$

This is the same across time. It also gets smaller as the periods get further apart if $|\phi| < 1$.

Notes

Correlation

Since $\text{Var}(y_t)$ is constant across time,

$$\gamma(s) = \text{Cor}(y_t, y_{t-s}) = \phi^s.$$

This is a function that is decreasing geometrically over time; this shape of the autocorrelation function is a distinguishing feature of the AR(1) model.

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Summary

Assuming that $|\phi| < 1$, an AR(1) process is covariance stationary and weakly dependent.

It is distinguished by the geometric decay in its autocorrelation function.

Though not discussed here, it is also characterized by having a *partial autocorrelation function* equal to ϕ at lag 1 and 0 for all other lags.

Notes